FI 3221 ELECTROMAGNETIC INTERACTIONS IN MATTER

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Physics of Magnetism and Photonics

SCATTERING OF LIGHT

- Rayleigh Scattering
- Scattering quantities
- Mie Scattering
WHAT IS SCATTERING?

- Imagine a particle under illumination of electromagnetic wave
• Electric charges inside the particle will experience oscillatory motion, and hence generate secondary EM wave

WHAT IS SCATTERING?

• This secondary wave is commonly known as scattered wave
Some part of the electromagnetic energy can be transformed into other forms (e.g., thermal energy).

This process is called **absorption**.

Rayleigh scattering applies when light is scattered by small particles.

By saying ‘small’ we mean particles with size much smaller than the wavelength of the incident light.
RAYLEIGH THEORY OF OPTICAL SCATTERING

- Small particles ($a << \lambda$): quasi-static approximation

The fields inside ($E_1$) and outside ($E_2$) the sphere may be found from the scalar potentials $\Phi_1(r, \theta)$ and $\Phi_2(r, \theta)$

$$E_1 = -\nabla \Phi_1 \quad E_2 = -\nabla \Phi_2$$

A SPHERE IN A UNIFORM STATIC FIELD: THE SOLUTION

- Laplace’s equations in source-free domains:

  \[
  \nabla^2 \Phi_1 = 0 \quad (r < a) \\
  \nabla^2 \Phi_2 = 0 \quad (r > a)
  \]

- Boundary conditions:

  $\Phi_1 = \Phi_2 \quad (r = a)$
  $\varepsilon_1 (\partial \Phi_1 / \partial r) = \varepsilon_2 (\partial \Phi_2 / \partial r) \quad (r = a)$
  $\partial \Phi_1 / \partial \theta = \partial \Phi_2 / \partial \theta \quad (r = a)$
  $\lim_{r \to \infty} \Phi_2 = -E_0 r \cos \theta$

  Solution:

  $\Phi_1 = -\frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \cdot E_0 r \cos \theta$
  $\Phi_2 = -E_0 r \cos \theta +$

  \[
  a^3 E_0 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \cdot \frac{\cos \theta}{r^2}
  \]
**A SPHERE IN A UNIFORM STATIC FIELD: THE SOLUTION**

- Electric potential inside and outside the sphere:
  \[ \Phi_1 = -\frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \cdot E_0 r \cos \theta \]
  \[ \Phi_2 = -E_0 r \cos \theta + a^3 E_0 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \cdot \frac{\cos \theta}{r^2} \]

- A sphere in an **static** field is equivalent to an ideal dipole
  - Dipole moment: \( p = \varepsilon_2 a E_0 \)
  - Dipole polarizability: \( \alpha = 4\pi a^3 \cdot \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \cdot 3V \)

**RAYLEIGH SCATTERING (a << \lambda)**

- Total scattered power (linearly polarized light):
  \[ P_s = \int_0^\pi 2\pi r^2 \sin \theta \cdot \frac{1}{2} E_x H^* \cdot d\theta = \frac{4\pi}{3} \sqrt{\varepsilon_2 \over \mu_0} \left( \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \right)^2 k^4 a^6 E_0^2 \]

- Scattering cross-section \( \sigma_s \) and scattering efficiency \( Q_s \):
  \[ \sigma_s = P_s \left( \frac{1}{2} \sqrt{\varepsilon_2 \over \mu_0} \cdot |E_0|^2 \right) = \frac{8\pi}{3} \left( \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \right)^2 k^4 a^6 \propto \alpha^2 \lambda^{-4} \]
  \[ Q_s = \sigma_s / \pi a^2 = \frac{8}{3} \left( \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \right)^2 k^4 a^4 \text{ (dimensionless number)} \]

- Short wavelength light is preferentially scattered
Scattering intensity of small particles is inversely proportional with the fourth power of wavelength of the incident light

\[ I_s \sim \frac{1}{\lambda^4} \]

This implies that for short wavelength the scattering will be strong

A similar derivation for a small cylinder yields

\[ I_s \sim \frac{1}{\lambda^3} \]

\[ Q_{sca} = \frac{8}{3} (ka)^4 \frac{|n^2 - 1|}{n^2 + 2} \]

\[ Q_{sca} = \frac{\pi^2}{4} (ka)^3 \frac{|n^2 - 1|}{n^2 + 1} \]

It explains why the sky is blue
How to analyse scattering by larger object?

Use integrated Poynting vector.

The fields in the region outside of the scatterer can be written as

\[ \vec{E}_2 = \vec{E}_i + \vec{E}_s \]
The Poynting vector in the region outside of the scatterer can be written as

\[ \mathbf{S} = \frac{1}{2} \text{Re} \left[ \mathbf{E} \times \mathbf{H}^* \right] \]
\[ = \frac{1}{2} \text{Re} \left[ (\mathbf{E}_i + \mathbf{E}_s) \times (\mathbf{H}_j + \mathbf{H}_s)^* \right] \]
\[ = \frac{1}{2} \text{Re} \left[ \mathbf{E}_j \times \mathbf{H}_j^* \right] + \frac{1}{2} \text{Re} \left[ \mathbf{E}_s \times \mathbf{H}_s^* \right] + \frac{1}{2} \text{Re} \left[ (\mathbf{E}_j \times \mathbf{H}_j^*) \times (\mathbf{E}_s + \mathbf{H}_s^*) \right] \]
\[ = \mathbf{S}_{inc} + \mathbf{S}_{sca} + \mathbf{S}_{ext} \]

The power absorb by the scatterer can be evaluated as

\[ W_{abs} = -\int_A \mathbf{S} \cdot d\mathbf{a} = W_{inc} - W_{sca} + W_{ext} \]
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\[
W_{abs} = \oint_{A} \vec{S} \cdot d\vec{a} = W_{inc} - W_{sca} + W_{ext}
\]
where
\[
W_{inc} = -\oint_{A} \vec{S}_{inc} \cdot d\vec{a}, \quad W_{sca} = \oint_{A} \vec{S}_{sca} \cdot d\vec{a}, \quad W_{ext} = -\oint_{A} \vec{S}_{ext} \cdot d\vec{a}
\]
with
\[
\vec{S}_{inc} = \frac{1}{2} \text{Re} \left[ \vec{E}_{i} \times \vec{H}_{s}^{*} \right]
\]
\[
\vec{S}_{sca} = \frac{1}{2} \text{Re} \left[ \vec{E}_{s} \times \vec{H}_{s}^{*} \right]
\]
\[
\vec{S}_{ext} = \frac{1}{2} \text{Re} \left[ \left( \vec{E}_{i} \times \vec{H}_{s}^{*} \right) \times \left( \vec{E}_{s} + \vec{H}_{s}^{*} \right) \right]
\]

Obviously, for a closed surface are, \( W_{inc} = -\oint_{A} \vec{S}_{inc} \cdot d\vec{a} = 0 \)
Obviously, for a closed surface area, \( W_{inc} = \int_A \vec{S}_{inc} \cdot d\vec{a} = 0 \)

Hence,

\[
W_{abs} = -W_{sca} + W_{ext}
\]

\[
W_{ext} = W_{sca} + W_{abs}
\]

A cross section is defined as the normalized power with respect to incoming wave, hence

\[
\sigma_{sca} = \frac{W_{sca}}{I_i}, \quad \sigma_{abs} = \frac{W_{abs}}{I_i}, \quad \sigma_{ext} = \frac{W_{ext}}{I_i}
\]

From this cross sections, we can define efficiencies as,

\[
\sigma_{sca} = A_{eff} Q_{sca}, \quad \sigma_{abs} = A_{eff} Q_{abs}, \quad \sigma_{ext} = A_{eff} Q_{ext}
\]
To find the exact wave solution, we write that all fields as a linear combination of some orthogonal base functions

\[ \vec{E}(\vec{r}) = \sum_{n} [a_n f_n^{(1)}(\vec{r})\hat{e}_1 + b_n f_n^{(2)}(\vec{r})\hat{e}_2 + c_n f_n^{(3)}(\vec{r})\hat{e}_3] \]

The orthogonal base function, \( f_n^{(i)}(\vec{r}) \), is chosen according to the (symmetry) of the problem. It could be a combination of

- spherical harmonics and spherical Bessel function for spherical symmetric scatterer,
- Bessel function for cylindrical symmetric scatterer.

\[ \mathbf{V} \times \mathbf{E} = \frac{1}{\rho} \begin{pmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ E_\rho & \rho E_\phi & E_z \end{pmatrix} = i \omega \mu (\hat{\rho}H_\rho + \hat{\phi}H_\phi + \hat{z}H_z) \]

\[ \mathbf{V} \times \mathbf{H} = \frac{1}{\rho} \begin{pmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{pmatrix} = -i \omega \varepsilon (\hat{\rho}E_\rho + \hat{\phi}E_\phi + \hat{z}E_z) \]
SCATTERING BY INFINITE CYLINDER

- Consider a problem as depicted in the figure
- A cylinder is illuminated by an electromagnetic plane wave
- The plane wave is propagating on x-y plane toward negative y-axis (no z-dependence)
- The magnetic field is polarized parallel with the z-axis

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SCATTERING BY INFINITE CYLINDER

- From the Maxwell equations, we obtain
  \[ H_z = \frac{1}{i \rho \omega \mu} \left[ \frac{\partial}{\partial \rho} (\rho E_\phi) - \frac{\partial E_\rho}{\partial \phi} \right] \]
  \[ E_\rho = -E_\phi = \frac{1}{i \omega \varepsilon} \left[ \frac{\partial H_z}{\partial \rho} \right] \]

- Solving for the magnetic field,
  \[ \rho^2 \frac{\partial^2 H_z}{\partial \rho^2} + \rho \frac{\partial H_z}{\partial \rho} + k^2 \rho^2 H_z = -\frac{\partial^2 H_z}{\partial \phi^2} \]
  with
  \[ k^2 = \mu \varepsilon \omega^2 \]

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Solving the previous partial differential equation by separation of variable method, the radial part yield

\[ \rho^2 \frac{\partial^2 R_z(k\rho)}{\partial \rho^2} + \rho \frac{\partial R_z(k\rho)}{\partial \rho} + (k^2 \rho^2 - \nu^2)R_z(k\rho) = 0 \]

Which is the Bessel equation, with solution can be a combination of:

1. Bessel function \( J_\nu(k\rho) \)
2. Neumann function \( Y_\nu(k\rho) \)
3. Hankel function of the first kind \( H_\nu(k\rho) \)
4. Hankel function of the second kind \( H^{(2)}_\nu(k\rho) \)

\[ \chi(\rho, \phi) = \sum_{\nu = -\infty}^{\infty} [M_\nu J_\nu(k\rho) + N_\nu H_\nu(k\rho)] e^{i\nu \phi} \]

\( \chi(\rho, \phi) \) can be the electric field or the magnetic field

\( M_\nu \) and \( N_\nu \) are the unknown coefficients yet to be determined
Since our problem is invariant in z direction, we may consider only the cross-sectional situation of the problem as depicted below.

In region 1, there is an incoming plane wave approaching a cylinder.

The cylinder will generate secondary electromagnetic field.
In region 2, a standing wave ensues

\[ \sum N_\nu H_\nu (k_1 \rho) e^{i \nu \phi} \]

\[ \sum M_\nu J_\nu (k_2 \rho) e^{i \nu \phi} \]

Applying the boundary conditions, we obtain the following value for the coefficients \( M_\nu \) and \( N_\nu \)

\[ N_\nu = (-1)^{\nu-1} \frac{n J'_\nu (k_1 a) J_\nu (k_2 a) - J_\nu (k_1 a) J'_\nu (k_2 a)}{n H'_\nu (k_1 a) J_\nu (k_2 a) - H_\nu (k_1 a) J'_\nu (k_2 a)} \]

\[ M_\nu = (-1)^{\nu} \frac{n J_\nu (k_1 a) H'_\nu (k_1 a) - n J'_\nu (k_1 a) H_\nu (k_1 a)}{n J_\nu (k_2 a) H'_\nu (k_1 a) - J'_\nu (k_2 a) H_\nu (k_1 a)} \]

where \( n = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} \)
INTENSITY AND POWER OF THE SCATTERED WAVE

- Intensity of the scattered wave can be derived from the definition of Poynting vector, more specifically, the time-averaged Poynting vector

\[ \mathbf{S}_{\text{sc}} = \frac{1}{2} \text{Re} \left\{ \mathbf{E}_{\text{sc}} \times \mathbf{H}_{\text{sc}}^{(*)} \right\} \]

thus, the radial component of the Poynting vector is

\[ S_{\rho}^{\text{sc}} = \frac{1}{2} \text{Re} \left\{ \mathbf{E}_{\phi 1}^{\text{sc}} \times \mathbf{H}_{z1}^{\text{sc}} \right\} \]

- Inserting the field components yield

\[ S_{\rho}^{\text{sc}} = \frac{1}{\omega \varepsilon_1} \frac{\pi a}{2} \sum \left| N_\nu \right|^2 \hat{\rho} \]

\[ P_{\text{sca}} = \int S_{\rho}^{\text{sc}} \cdot dA \]

or,

\[ P_{\text{sca}} = \int \frac{1}{\omega \varepsilon_1 \pi a} \sum \left| N_\nu \right|^2 \hat{\rho} \cdot RL \ d\phi \hat{\rho} \]

or,

\[ P_{\text{sca}} = \frac{2L}{\omega \varepsilon_1} \sum \left| N_\nu \right|^2 \]
Scattering Cross Section ($C_{sca}$) is the ratio of $P_{sca}$ with respect to the intensity of the incoming plane wave

$$C_{sca} = \frac{P_{sca}}{I_{inc}}$$

Intensity of an incoming plane wave with the amplitude of its magnetic field is unity can be written as

$$I_{inc} = \frac{\mu_0 c}{2}$$

Thus

$$C_{sca} = \frac{4L}{k_1} \sum N_v |N_v|^2$$

Scattering Efficiency ($Q_{sca}$) is defined as

$$Q_{sca} = \frac{C_{sca}}{G}$$

$G$ is the particle cross-sectional area as viewed by the incoming wave, thus for our case $G = 2aL$

This yields

$$Q_{sca} = \frac{2}{k_1 a} \sum |N_v|^2$$
EXAMPLES

• Silver nanocylinder is illuminated by EM plane wave

![Graph showing the response of silver nanocylinders to EM plane waves at different sizes and wavelengths.]

• Silver nanosphere is illuminated by EM plane wave

![Graph showing the response of silver nanospheres to EM plane waves at different sizes and wavelengths.]

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APPLICATIONS

\[ \varepsilon_2 = \text{silver} \]

\[ \varepsilon_1 = \text{silver} \]

75 nm

100 nm

\[ O^2 \]

\[ eV \]

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